Matérn-based nonstationary cross-covariance models for global processes

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Outline

1 Objective

2 Background

3 Model

4 Application
Cross-covariance models

- Suppose \( \mathbf{Z}(\mathbf{s}) = \{Z_1(\mathbf{s}), \ldots, Z_N(\mathbf{s})\} \) is a multivariate stochastic process defined on \( \mathbf{s} \in D \subset \mathcal{S}^2 \) or \( \mathbb{R}^3 \).

- We let \( Z_i(\mathbf{s}) = \mu_i(\mathbf{s}) + e_i(\mathbf{s}) \), \( \mu_i \) is assumed to be fixed, and \( e_i \) is mean zero, \( e_i \)'s are jointly normal.

- We are interested in modeling \( \text{Cov}\{e_i(\mathbf{s}_1), e_j(\mathbf{s}_2)\} \) as a parametric function depending on \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \).

- There is an issue about positive definiteness for the matrix-valued function:

\[
A(\mathbf{s}_1, \mathbf{s}_2) = \begin{pmatrix}
\text{Cov}\{e_i(\mathbf{s}_1), e_i(\mathbf{s}_2)\} & \text{Cov}\{e_i(\mathbf{s}_1), e_j(\mathbf{s}_2)\} \\
\text{Cov}\{e_j(\mathbf{s}_1), e_i(\mathbf{s}_2)\} & \text{Cov}\{e_j(\mathbf{s}_1), e_j(\mathbf{s}_2)\}
\end{pmatrix}
\]

Matérn class on a sphere:

A Matérn class is given by

$$K(d) = \frac{\alpha}{2^{\nu-1}\Gamma(\nu)}\left(\frac{d}{\beta}\right)^\nu K_\nu\left(\frac{d}{\beta}\right), \quad d = |\mathbf{s}_1 - \mathbf{s}_2|$$

Matérn class is positive definite if and only if $\nu \leq 0.5$ with great circle distance (Miller and Samko 2001, Gneiting 2013)
Then what do we do on the sphere?

“Easy” solution (Yadrenko 1983): projection of covariance function on $\mathbb{R}^3 \times \mathbb{R}^3$ to $S^2 \times S^2$ through

$$r = 2R \sin(\theta/2)$$

There are recent work on other approaches (Guinness and Fuentes 2013, Jeong and Jun 2013, Heaton et al 2013, Du et al 2013 among others)

But what do we do when the processes are nonstationary?
Nonstationary covariance models on a sphere

- My interest is in developing nonstationary covariance models for multivariate processes on a sphere.

- Suppose we have $Z_i(L_j, l_j)$ with $L_j$ latitude and $l_j$ longitude and we consider $\text{Cov}\{Z_1(L_1, l_1), Z_2(L_2, l_2)\} = K(L_1, L_2, l_1, l_2)$.

- Particular types of nonstationarity to consider for global data:
  - Dependence of (co)variance on latitude.
  - Dependence on altitude, or land/sea.
  - Longitudinal irreversibility: $\text{Cov}\{Z_1(L_1, l_1), Z_2(L_2, l_2)\} \neq \text{Cov}\{Z_1(L_1, l_2), Z_2(L_2, l_1)\}$ for some $L_i, l_i$.
  - Asymmetry for multivariate case: $\text{Cov}\{Z_1(L_1, l_1), Z_2(L_2, l_2)\} \neq \text{Cov}\{Z_1(L_2, l_2), Z_2(L_1, l_1)\}$ for some $L_i$ and $l_i$. 
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Isotropic multivariate Matérn function

- Gneiting et al (2010, JASA)

- $K_{ij}(s_1, s_2) = \text{cov}\{Z_i(s_1), Z_j(s_2)\}$ and $s_k = (L_k, l_k)$

- $K_{ii}(s_1, s_2) = \frac{\sigma_i^2}{2^{\nu_i-1}\Gamma(\nu_i)} M_{\nu_i}(d(s_1, s_2)/\xi_i),$ 

- $K_{ij}(s_1, s_2) = \rho_{ij} \frac{\sigma_i \sigma_j}{2^{\nu_{ij}-1}\Gamma(\nu_{ij})} M_{\nu_{ij}}(d(s_1, s_2)/\xi_{ij})$

- $M_{\nu}(x) = x^\nu K_{\nu}(x)$

- Some restrictive conditions necessary for $\nu_{ij}, \xi_{ij}, \rho_{ij}$
Isotropic multivariate Matérn function

Covariance: Pressure

Covariance: Temperature

Cross Covariance
Nonstationary multivariate Matérn function


- $K_{ii}(s_1, s_2) = \sigma_i(L_1)\sigma_i(L_2)\mathcal{M}_{v_{ii}}(s_1, s_2)(d(s_1, s_2)/\xi_i)$,

- $K_{ij}(s_1, s_2) = \beta_{ij}\sigma_i(L_1)\sigma_j(L_2)\mathcal{M}_{v_{ij}}(s_1, s_2)(d(s_1, s_2)/\xi_{ij})$

- Note $\beta_{ij}$ is constant over space

- Kleiber and Genton (2012) for post processing of cross-correlation
Differential operators approach

- Jun (2011, SJS)


\[ Z_i(L, l) = \left\{ A_i(L) \frac{\partial}{\partial L} + B_i(L) \frac{\partial}{\partial l} \right\} Y(L, l) \]

- Covariance of \( Y \) described by isotropic Matérn covariance model

- The coefficients are modeled through linear combinations of Legendre polynomials, for example:

\[ A_i(L) = \sum_{j=0}^{m} a_{ij} P_j(\sin L), \]
The approach in Jun (2011)

- Unrealistic cross correlation near the poles
  \[ \text{Cor}\{Z_i(L_1, l_1), Z_j(L_2, l_2)\} \to 1 \]
  as \( L_1, L_2 \to \pm \pi/2 \)

- Differential operators do not respect geometry of the sphere

- Singularity at the poles:
  - For the model to be \textit{mean square continuous} at the poles (in \( L_2 \) sense), it needs to satisfy (Jun and Stein, 2007)
    \[
    \lim_{L \to \pm \pi/2} \left\{ A^2(L) + B^2(L) \cos^2 L \right\} = 0
    \]

- Nonstationarity due to spatially dependent factor (e.g. land/sea)?
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What we propose...

\[ Z_i(L, l) = \left\{ A_i(L) \frac{\partial}{\partial L} + B_i(L) \frac{\partial}{\partial l} \right\} Y_i(L, l) \]

I. For \( Y_i \), we use nonstationary multivariate Matérn model by Kleiber and Nychka (2012):

- In particular, we let \( \nu_i \)'s vary over altitude or depend on latitude
- \( \xi_i \)'s fixed over space
- \( \sigma_i \)'s vary over altitude and latitude

\[ \text{Cor}\{Z_i(L_1, l_1), Z_j(L_2, l_2)\} \rightarrow \beta_{ij} \text{ as } L_1, L_2 \rightarrow \pm \pi/2 \]
What we propose...

\[ Z_i(L, l) = \left\{ A_i(L) \frac{\partial}{\partial L} + B_i(L) \frac{\partial}{\partial l} \right\} Y_i(L, l) \]

II. For \( A_i \) and \( B_i \), we consider

1. \( A_i(L) = P(a_{i,0}, \ldots, a_{i,p}; L), \quad B_i(L) = 1/\cos L P(b_{i,0}, \ldots, b_{i,q}; L) \)

OR

2. \( A_i(L) = \cos L P(a_{i,0}, \ldots, a_{i,p}; L), \quad B_i(L) = P(b_{i,0}, \ldots, b_{i,q}; L) \)
   (mean-square continuous at the poles, Hitzenko and Stein (2012))
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Two climate data sets: data 1

- 5 months average for Northern winter (November - March) of surface temperature and precipitation data averaged over 1970-1999

- Outputs from NCAR CCSM3 (CMIP3)

- Original resolution $256 \times 128$, we randomly sampled 1000 locations for model fit, 300 locations for prediction (100 locations common for both variables, and additional 100 locations each for each variable)
Two climate data sets: data 1

Surface temperature and precipitation data. Each process is separately mean filtered through regression with spherical harmonics terms as in Jun (2011). Pixels without circles denote the locations of \( \mathcal{L} \) and pixels with circles denote locations of \( \mathcal{L}^{(i)} \) \( (i = 1 \text{ for Temperature and } i = 2 \text{ for Precipitation}) \)
Two climate data sets: data 2

- Climate model “errors” from Geophysical Fluid Dynamics Laboratory (GFDL-CM2.0) and Hadley Center for Climate Prediction and Research (UKMO-HadCM3)

- Boreal winter average (December - February) averaged over 1970-1999 of surface temperature

- Jun, Knutti, Nychka (2008, JASA, Tellus) showed significant cross dependence between the two model errors

- Same spatial random sampling as data 1
Map of two climate model errors. Pixels without circles denote the locations of $\mathcal{L}$ and pixels with circles denote locations of $\mathcal{L}^{(i)}$ ($i = 1$ for GFDL-CM2.0 and $i = 2$ for UKMO-HadCM3).
List of covariance models

- IM: independent Matérn model (no cross-dependence)
- M1: bivariate isotropic Matérn model
- M2-x: nonstationary Matérn model from Kleiber and Nychka (2012)
- M3-x: combination of M2-x and differential operators approach
- M4-x: differential operators approach only
Comparison regarding model fit

### Application 1:

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<tr>
<th></th>
<th>M1</th>
<th>M2-2</th>
<th>M2-3</th>
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### Application 2:

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## Comparison regarding prediction I

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## Comparison regarding prediction II

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Thank you!